CHAPTER 6. THE EQUATION $x + y = z$ IN S-INTEGERS.

The results of this chapter have been published in de Weger [1987[a]].

6.1. Introduction.

Let S be the set of all positive integers composed of primes from a fixed finite set $\{ p_1, \ldots, p_s \}$, where $s \geq 3$. This chapter is devoted to the diophantine equation

$$x + y = z \tag{6.1}$$

in $x, y, z \in S$. Without loss of generality we may assume that x, y, z are relatively prime. For any $a \in S$ we define

$$m(a) = \max_{1 \leq i \leq s} \mathrm{ord}_{p_i}(a) .$$

It was proved by Mahler [1933] that (6.1) has only finitely many solutions, but his proof is ineffective. An effective version, i.e. an effectively computable upper bound for $m(x \cdot y \cdot z)$ for the solutions x, y, z of (6.1), can be derived from the results of Coates [1969], [1970] and Sprindžuk [1969], since (6.1) can be reduced to a finite number of Thue equations. See also Chapter 1 of Shorey and Tijdeman [1986].

We derive an explicit upper bound in Section 6.2. Section 6.3 is devoted to some details of the p-adic approximation lattices on which the reduction method of Sections 6.4 and 6.5 are based. In Section 6.4 we give a method of solving (6.1) in the one-dimensional case $s = 3$. This method is based on the reduction procedure given in Section 3.10. As an example we find all the solutions of the slightly more general equation $x \pm y = w \cdot z$, where x, y, z are powers of 2, 3 or 5, and $w \in \mathbb{Z}$, $|w| \leq 1000000$, $(w,z) = 1$. In Section 6.5 we give a procedure for solving (6.1) in the multi-dimensional case $s \geq 4$, based on the reduction procedure described in Section 3.11. We work out the example $\{ p_1, \ldots, p_6 \} = \{ 2, 3, 5, 7, 11, 13 \}$, and actually determine all the solutions. This generalizes the result of Alex [1976], who gave a complete solution of (6.1) for the case $\{ p_1, \ldots, p_4 \} = \{ 2, 3, 5, 7 \}$ by elementary arguments. See also Rumsey and Posner [1964] and Brenner and Foster [1982]. We conclude in Section 6.6 with some remarks on the Oesterlé–Masser conjecture, also known as the 'abc'-conjecture, which is related to equation (6.1). In particular, our method of solving (6.1) leads to a method of finding examples that are of interest with respect to the abc-conjecture. Finally, we give tables in Section 6.7.

6.2. Upper bounds.

We give in this section an upper bound for the solutions of (6.1), based on lemma 2.6 (cf. Yu [1987[a]]). Note that in our paper de Weger [1987[a]] we used the result of van der Poorten [1977] instead of Yu's.

We introduce a lot of notation. Assume that $p_1 < \ldots < p_s$. Let q_i be the smallest prime with $q_i \nmid p_i \cdot (p_i - 1)$ for $i = 1, \ldots, s$. Put

$$t = [2 \cdot s/3] , \quad P = \prod_{i=1}^{s} p_i , \quad q = \max_i q_i ,$$

$C_1(2,t)$ and a_1 as in lemma 2.6 with $n = t$,

$$U = C_1(2,t) \cdot a_1^t \cdot t^{t+5/2} \cdot q^{2 \cdot t} \cdot (q-1) \cdot \log^2(t \cdot q) \cdot \max_i \frac{(p_i - 1) \cdot \left(2 + \frac{1}{p_i - 1}\right)^t}{(\log p_i)^{t+2}} \cdot$$

$$\cdot (\log p_s)^t \cdot \left(\log(4 \cdot \log p_s) + \frac{\log p_s}{8 \cdot t} \right) ,$$

$$C_1 = U/6 \cdot t , \quad C_2 = U \cdot \log 4 ,$$

$$V_i = \max(1, \log p_i) \quad \text{for} \quad i = s-t+1, \ldots, s , \quad \Omega = \prod_{i=s-t+1}^{s} V_i ,$$

$$C_3 = 2^{9 \cdot t + 26} \cdot t^{t+4} \cdot \Omega \cdot \log(e \cdot V_{s-1}) ,$$

$$C_4 = \max \left(7.4, \left(C_1 \cdot \log(P/p_1) + C_3\right)/\log p_1 \right) ,$$

$$C_5 = \left(C_2 \cdot \log(P/p_1) + C_3 \cdot \log(e \cdot V_s) + 0.327\right)/\log p_1 ,$$

$$C_6 = \max \left(C_5, (C_2 \cdot \log(P/p_1) + \log 2)/\log p_1 \right) ,$$

$$C_7 = 2 \cdot \left(C_6 + C_4 \cdot \log C_4 \right) ,$$

$$C_8 = \max\left(p_s,\ \log\left[2\cdot(P/p_1)^{p_s}\right]/\log p_1,\ C_2+C_1\cdot\log C_7,\ C_7 \right) .$$

Now we state the main result.

<u>THEOREM 6.1.</u> *The solutions of* (6.1) *satisfy* $m(x\cdot y\cdot z) \le C_8$.

<u>Proof.</u> If we consider instead of (6.1) the equivalent equation

$$x \pm y = z \tag{6.2}$$

then we may assume that $x\cdot y$ has at most t prime divisors, $p_{i_1}, \ldots, p_{i_t}$ say. Suppose first that $m(x\cdot y) \le p_s$. Then

$$p_1^{m(z)} \le z \le 2\cdot\max(x,y) < 2\cdot(P/p_1)^{p_s} ,$$

hence

$$m(x\cdot y\cdot z) < \max\left(p_s,\ \log\left[2\cdot(P/p_1)^{p_s}\right]/\log p_1 \right) \le C_8 .$$

Next suppose that $m(x\cdot y) \ge p_s$ and $m(z) \ge 2$. Then for some $p = p_i$,

$$m(z) = \mathrm{ord}_p(z) = \mathrm{ord}_p\left(\pm\frac{x}{y} - 1 \right) = \mathrm{ord}_p\left(\log_p\left(\frac{x}{y}\right)\right) .$$

Put $x/y = \prod_{j=1}^{t} p_{i_j}^{x_{i_j}}$. Then $m(x\cdot y) = \max_{1\le j\le t} |x_{i_j}|$. We apply Lemma 2.6 (Yu's lemma) with $n = t,\ B_0 = B_n = B' = B = m(x\cdot y)$. Since $m(x\cdot y) \ge p_s$ and $t \ge 2$ we have

$$W = \max\left(\log\left(1+\frac{3}{4\cdot t}\cdot B\right),\ \log B,\ \log p \right) = \log B .$$

Note that $C_1(p,n)$ is maximal for $p = 2$. We obtain

$$m(z) < C_1\cdot\log m(x\cdot y) + C_2 . \tag{6.3}$$

Obviously (6.3) is also true if $m(z) < 2$. If in (6.2) the plus sign holds, then

$$(P/p_1)^{m(z)} \ge z > \max(x,y) \ge p_1^{m(x\cdot y)} .$$

By (6.3) and $C_3 > 0$ it then follows that

$$m(x\cdot y) < C_4\cdot\log m(x\cdot y) + C_6 \ . \tag{6.4}$$

Next suppose that in (6.2) the minus sign holds. Then we apply Lemma 2.4 to prove (6.4) for this case, as follows. Suppose (6.4) is false. Then

$$\left| \frac{y}{x} - 1 \right| = \frac{z}{x} = \frac{z}{\max(x,y)} \le \frac{(P/p_1)^{m(z)}}{p_1^{m(x\cdot y)}} < \frac{(P/p_1)^{C_1\cdot\log m(x\cdot y)+C_2}}{p_1^{C_4\cdot\log m(x\cdot y)+C_6}} ,$$

which is less than $\frac{1}{2}$, by the definition of C_4 and C_6. Hence

$$\left|\log \frac{y}{x}\right| < (2\cdot\log 2)\cdot\left| \frac{y}{x} - 1 \right| < (2\cdot\log 2)\cdot\frac{(P/p_1)^{C_1\cdot\log m(x\cdot y)+C_2}}{p_1^{m(x\cdot y)}} .$$

On the other hand, Lemma 2.4 yields

$$\left|\log \frac{y}{x}\right| > \exp\left(-C_3\cdot(\log m(x\cdot y) + \log(e\cdot V_s)) \right) .$$

Thus we obtain

$$m(x\cdot y)\cdot\log p_1 < \log(2\cdot\log 2) + (C_1\cdot\log m(x\cdot y)+C_2)\cdot\log(P/p_1)$$
$$+ C_3\cdot(\log m(x\cdot y) + \log(e\cdot V_s)) \le (\log p_1)\cdot(C_4\cdot\log m(x\cdot y)+C_6) .$$

This contradicts our assumption that (6.4) if false. Consequently (6.4) is true in all cases. Now, by $C_4 > e^2$, Lemma 2.1 yields $m(x\cdot y) < C_7$, and (6.3) then yields $m(x\cdot y\cdot z) < C_8$. □

<u>Examples.</u> If $s = 3$, $\{ p_1, p_2, p_3 \} = \{ 2, 3, 5 \}$ then $C_8 < 3.98\times10^{17}$. If $s = 6$, $\{ p_1, \ldots, p_6 \} = \{ 2, 3, 5, 7, 11, 13 \}$ then $C_8 < 5.60\times10^{27}$.

6.3. The p-adic approximation lattices.

As in the proof of Theorem 6.1 we consider (6.2) instead of (6.1). Let p be any of the primes $p_1, \ldots, p_s$. We may assume that $p \nmid x\cdot y$. Rename the other primes as $p_0, \ldots, p_{s-2}$, such that $\mathrm{ord}_p(\log_p(p_0))$ is minimal. For $i = 1, \ldots, s-2$ put (cf. Section 3.11)

$$\vartheta_i = -\log_p(p_i)/\log_p(p_0) = \sum_{\ell=0}^{\infty} u_{i,\ell}\cdot p^{\ell} ,$$

where $u_{i,\ell} \in \{ 0, 1, \ldots, p-1 \}$. The ϑ_i take the place of the ϑ'_i of Section 3.11. Then it is clear from Section 3.11 how to define the p-adic approximation lattices Γ_μ for $\mu \in \mathbb{N}_0$. Put

$$\Lambda = \sum_{i=1}^{s-2} x_i \cdot \vartheta_i - x_0 .$$

Then Lemma 3.13 yields

$$\Gamma_\mu = \{ (x_1, \ldots, x_{s-2}, x_0) \mid |\Lambda|_p \le p^{-\mu} \}$$

$$= \{ (x_1, \ldots, x_{s-2}, x_0) \mid |\log_p\left(\prod_{i=0}^{s-2} p_i^{x_i}\right)|_p \le p^{-(\mu+\mu_0)} \} ,$$

where $\mu_0 = \mathrm{ord}_p(\log_p(p_0))$. In Section 3.13 we studied the set

$$\Gamma_\mu^* = \{ (x_1, \ldots, x_{s-2}, x_0) \mid |\prod_{i=0}^{s-2} p_i^{x_i} \pm 1|_p \le p^{-(\mu+\mu_0)} \} ,$$

which is a sublattice of Γ_μ . In Lemma 3.17 we showed how a basis of Γ_μ^* can be found from a basis of Γ_μ . In practice this is very easy, especially if for $p \ge 5$ it happens to be possible to choose p_0 such that not only $\mathrm{ord}_p(\log_p(p_0))$ is minimal, but also p_0 is a primitive root (mod p) . Then, using the notation of Lemma 3.17 (with $\underline{b}_0$ as the last element of the basis), choose $\zeta \equiv p_0 \pmod p$. Then $k(\underline{b}_0) = 1$, and it follows that $\underline{b}'_i = \underline{b}_i$ for $i = 1, \ldots, s-2$. By $\underline{b}_i = \left(0, \ldots, 1, \ldots, 0, \vartheta_i^{(\mu)}\right)^T$ we have

$$p_i \cdot p_0^{\vartheta_i^{(\mu)}} \equiv \zeta^{k(\underline{b}_i)} \pmod{p^{\mu+\mu_0}} .$$

If $p_i \equiv p_0^{\alpha_i} \pmod p$, then it follows that

$$\gamma_i^* \equiv \alpha_i + \vartheta_i^{(\mu)} \equiv \alpha_i + \sum_{\ell=0}^{\mu-1} u_{i,\ell} \pmod{(p-1)/2} \quad \text{for } i = 1, \ldots, s-2 ,$$

$$\gamma_0^* = (p-1)/2 .$$

Lemma 3.14 (with $c_1 = 0$, $c_2 = 1$) now yields: if

$$\ell(\Gamma_\mu^*) > \sqrt{(s-1)} \cdot X_1 \qquad (6.5)$$

then (6.2) has no solutions with

$$\mu + \mu_0 \le \mathrm{ord}_p(z) \le m(x \cdot y \cdot z) \le X_1 . \qquad (6.6)$$

6.4. Reducing the upper bounds in the one-dimensional case.

In Section 3.10 we have described how an upper bound for the solutions of (6.1) in the case $s = 3$ can be reduced. We shall apply that method in this section to the following problem.

THEOREM 6.2. *The diophantine equation*

$$x \pm y = w \cdot z \ , \tag{6.7}$$

where $x = p_0^{x_0}$, $y = p_1^{x_1}$, $z = p^u$, $(p, p_0, p_1) = (2,3,5), (3,2,5)$ *or* $(5,2,3)$, $x_0, x_1, u \in \mathbb{N}_0$, $w \in \mathbb{Z}$, $|w| \le 10^6$, *and* $p \nmid w$, *has exactly* 291 *solutions for* $p = 2$, 412 *solutions for* $p = 3$, *and* 570 *solutions for* $p = 5$. *In* Table I *all solutions with* $u \ge 3$ *are given. The solutions with* $u \le 2$ *satisfy* $x_0 \le 14$, $x_1 \le 9$ *for* $p = 2$, $x_0 \le 23$, $x_1 \le 10$ *for* $p = 3$, *and* $x_0 \le 25$, $x_1 \le 15$ *for* $p = 5$.

Remark. It is easy to find all solutions of (6.7) with $u \le 2$. The Tables are presented in Section 6.7.

Proof. Put $X = \max_{p=2,3,5} \operatorname{ord}_p(x \cdot y \cdot z)$. The example at the end of Section 6.2 shows that in the case $|w| = 1$ we have $X < 3.98 \times 10^{17}$. It can be checked without difficulties that the effect of the w with $|w| \le 10^6$ in the proof of Theorem 6.1 can be neclected (it disappears in the rounding off), so that for the solutions of (6.7) also $X < X_0 = 3.98 \times 10^{17}$ holds. Put

$$x/y = p_0^{y_0} \cdot p_1^{y_1} \ , \quad \vartheta = -\log_p(p_1)/\log_p(p_0) \ .$$

Note that ϑ is a p-adic integer. Define the lattices Γ_μ, Γ_μ^* as in Section 6.3, so Γ_μ is generated by

$$\underline{b}_1 = \begin{pmatrix} 1 \\ \vartheta^{(\mu)} \end{pmatrix} , \quad \underline{b}_0 = \begin{pmatrix} 0 \\ p^\mu \end{pmatrix} .$$

For $p = 2, 3$ we have $\Gamma_\mu^* = \Gamma_\mu$, and for $p = 5$ a basis of Γ_μ^* is

$$\underline{b}_1^* = \underline{b}_1 - \gamma \cdot \underline{b}_0 \ , \quad \underline{b}_0^* = 2 \cdot \underline{b}_0 \ ,$$

where $\gamma = 0$ if $\vartheta^{(\mu)}$ is odd, $\gamma = 1$ if $\vartheta^{(\mu)}$ is even . Using the algorithm given in Section 3.10, Fig. 3, we can compute a basis $\underline{c}_1$, $\underline{c}_2$ of Γ_μ^* that is reduced in the sense that $|\underline{c}_1| = \ell(\Gamma_\mu^*)$. We did so, with μ as in the following table.

p	p_0	p_1	μ_0	μ	γ	$\lvert\underline{c}_1\rvert >$	$u \le$	W	$\lvert y_0\rvert \le$	$\lvert y_1\rvert \le$
2	3	5	2	143		2.68×10^{21}	144	$10^6\cdot2^{144}$	114	78
3	2	5	1	91		2.32×10^{21}	91	$10^6\cdot3^{91}$	182	78
5	2	3	1	65	0	5.28×10^{22}	65	$10^6\cdot5^{65}$	189	119

The values of $\vartheta^{(\mu)}$ can be found in Table III. Making an exception to our policy, we give the reduced bases of the Γ_μ^* below.

p = 2 :

$$\underline{c}_1 = \left(\begin{array}{r} 10\ 00000\ 00100\ 10001\ 10110\ 01110\ 01101 \\ 00001\ 11101\ 00101\ 00100\ 11100\ 01111\ 11010\ 00011 \\ \\ -\ 1\ 00010\ 00110\ 01000\ 01011\ 01110\ 00010 \\ 00101\ 11000\ 00000\ 11100\ 01111\ 01011\ 10111\ 00001 \end{array}\right),$$

$$\underline{c}_2 = \left(\begin{array}{r} 10\ 11011\ 10000\ 01011\ 01101\ 11000\ 00111 \\ 11001\ 10100\ 11011\ 00000\ 11111\ 10110\ 10110\ 00001 \\ \\ 10\ 01110\ 11101\ 10111\ 11000\ 00100\ 10101 \\ 00111\ 00001\ 10101\ 00110\ 10011\ 00111\ 00101\ 10101 \end{array}\right),$$

p = 3 :

$$\underline{c}_1 = \left(\begin{array}{r} -\ 102\ 01121\ 02221\ 00210\ 12120\ 20020\ 22222\ 10212\ 20222 \\ 21002\ 00122\ 21100\ 11102\ 22102\ 20001\ 11222\ 02212\ 21011 \end{array}\right),$$

$$\underline{c}_2 = \left(\begin{array}{r} -10\ 12210\ 12111\ 01102\ 02010\ 12112\ 12210\ 21122\ 21011\ 20102 \\ -\ 2\ 22021\ 11012\ 01000\ 12021\ 00211\ 12221\ 22121\ 21220\ 12122 \end{array}\right),$$

p = 5 :

$$\underline{c}_1 = \left(\begin{array}{r} -\ 211\ 32230\ 21042\ 22023\ 30141\ 33034\ 21420 \\ -\ 22104\ 43102\ 43111\ 03114\ 30134\ 23410 \end{array}\right),$$

$$\underline{c}_2 = \left(\begin{array}{r} 340\ 34003\ 02404\ 12120\ 03412\ 22030\ 32211 \\ -\ 414\ 20001\ 42202\ 42210\ 34043\ 20120\ 00432 \end{array}\right).$$

From this we found the lower bounds for $\lvert\underline{c}_1\rvert$ given above. They are all larger than $\sqrt{2}\cdot3.98\times10^{17}$. Hence (6.5) holds for $X_1 = X_0$, and then we infer from (6.6) that $u \le \mu + \mu_0 - 1$, and $\lvert w\rvert\cdot z \le W$ as shown in the table above. We now find the new upper bounds for $\lvert y_0\rvert$, $\lvert y_1\rvert$ as follows. If in (6.7) the minus sign holds, then, on supposing that $\min(x,y) > W^{10/9}$, we infer

$$| x - y | = |w| \cdot z \le W < \min(x,y)^{0.9} .$$

By Theorem 5.2(a), the inequality $| x - y | < \min(x,y)^{0.9}$ has no solutions with $\min(x,y) > W$, since $W > 10^{49}$. Hence $\min(x,y) \le W^{10/9}$, and we infer

$$\max(x,y) \le \min(x,y) + |w| \cdot z \le W^{10/9} + W .$$

If in (6.7) the plussign holds, then this inequality follows at once. So now the bounds given in the above table for $|y_0|$, $|y_1|$ follow from

$$|y_i| \cdot \log p_i \le \log \max(x,y) \le \log(W^{10/9}+W) .$$

We repeat the procedure with μ as in the following table.

p	μ	γ	$\lvert \underline{c}_1 \rvert >$	$\sqrt{2} \cdot X_0 <$	$u \le$	W	$\lvert y_0 \rvert \le$	$\lvert y_1 \rvert \le$
2	16		167.7	161.3	17	$10^6 \cdot 2^{17}$	31	21
3	13		535.8	257.4	13	$10^6 \cdot 3^{13}$	49	21
5	7	1	276.1	267.3	7	$10^6 \cdot 5^7$	49	31

The numbers are now so small that the computations can be performed by hand. For example, for $p = 5$, the lattice Γ_7^* is generated by

$$\underline{b}_1^* = \begin{pmatrix} 1 \\ -45607 \end{pmatrix} , \quad \underline{b}_0^* = \begin{pmatrix} 0 \\ 156250 \end{pmatrix} ,$$

and a reduced basis is

$$\underline{c}_1 = \begin{pmatrix} 185 \\ 205 \end{pmatrix} , \quad \underline{c}_0 = \begin{pmatrix} -394 \\ 408 \end{pmatrix} .$$

We find upper bounds for u and W as given in the above table. In all three cases, $W^{10/9} < 10^{15}$. On supposing $\min(x,y) > 10^{15}$ we infer

$$| x - y | = |w| \cdot z \le W < 10^{15 \cdot 0.9} \le \min(x,y)^{0.9} .$$

By Theorem 5.2(a) we see that the inequality $| x - y | < \min(x,y)^{0.9}$ has only two solutions: $(x,y) = (2^{65},5^{28}), (2^{84},3^{53})$. However, both have $| x - y | > 10^{15 \cdot 0.9}$. So we infer $\min(x,y) \le 10^{15}$, hence by $\max(x,y) \le 10^{15} + W$ we obtain the bounds for $|y_0|$, $|y_1|$ as given above. These bounds are small enough to admit enumereation of the remaining cases. □

Remark. The computer calculations for the above proof took less than 1 sec.

6.5. Reducing the upper bounds in the multi-dimensional case.

In Section 3.11 we have described how an upper bound for the solutions of (6.1) in the case $s \geq 3$ can be reduced. We shall apply that method in this section to the following problem.

THEOREM 6.3. *The diophantine equation*

$$x + y = z \tag{6.8}$$

in $x, y, z \in S = \{ 2^{x_1} \cdot \ldots \cdot 13^{x_6} \mid x_i \in \mathbb{N}_0 \text{ for } i = 1, \ldots, 6 \}$ *with* $(x,y) = 1$ *and* $x \leq y$ *has exactly* 545 *solutions. Of them,* 514 *satisfy*

$$ord_2(x \cdot y \cdot z) \leq 12 , \quad ord_3(x \cdot y \cdot z) \leq 7 , \quad ord_5(x \cdot y \cdot z) \leq 5 ,$$

$$ord_7(x \cdot y \cdot z) \leq 4 , \quad ord_{11}(x \cdot y \cdot z) \leq 3 , \quad ord_{13}(x \cdot y \cdot z) \leq 4 .$$

The remaining 31 *solutions are given in* Table II.

Remark. From Theorem 6.3 it is not much effort to find all 545 solutions of (6.8).

Proof. In the example at the end of Section 6.2 we have seen that $m(x \cdot y \cdot z) < X_0 = 5.60 \times 10^{27}$. With the notation of Section 6.3 we choose the following parameters.

p	p_0	p_1	p_2	p_3	p_4	μ_0	μ	γ_0^*	γ_1^*	γ_2^*	γ_3^*	γ_4^*
2	3	5	7	11	13	2	605					
3	2	5	7	11	13	1	385					
5	2	3	7	11	13	1	275	2	0	1	1	1
7	3	2	5	11	13	1	220	3	0	1	1	0
11	2	3	5	7	13	1	165	5	−2	0	1	1
13	2	3	5	7	11	1	165	6	2	1	2	3

We computed the six values of the $\vartheta_i^{(\mu)}$ for $i = 1, 2, 3, 4$ (and give them in Table III), and the reduced bases of the six lattices Γ_μ^*, by the L^3-algorithm. Thus we obtained:

p	$\ell(\Gamma_\mu^*) \geq \lvert \underline{c}_1 \rvert /4 >$	$ord_p(x\cdot y\cdot z) \leq$
2	4.70×10^{35}	606
3	1.15×10^{36}	385
5	6.27×10^{37}	275
7	3.17×10^{36}	220
11	5.74×10^{33}	165
13	1.73×10^{36}	165

These lower bounds for $\ell(\Gamma_\mu^*)$ are all larger than $\sqrt{5}\cdot 5.60\times10^{27}$ (note that we have a very large margin here, we could have taken the μ's probably about 20% smaller). So we apply Lemma 3.14 for $X_1 = X_0 = 5.60\times10^{27}$. For every p we thus find $ord_p(z) \leq \mu + \mu_0$. Since equation (6.2) is invariant under permutations of x, y, z , we even have $ord_p(x\cdot y\cdot z) \leq \mu + \mu_0$, as shown in the above table. Hence $m(x\cdot y\cdot z) \leq 606$.
We repeated the procedure with $X_0 = 606$ and μ as in the following table. After computing the reduced bases of the six lattices Γ_μ^* we found the following data. Note that in all cases $\ell(\Gamma_\mu^*) \geq \sqrt{5}\cdot 606$.

p	μ	γ_0^*	γ_1^*	γ_2^*	γ_3^*	γ_4^*	$\ell(\Gamma_\mu^*) >$	$ord_p(x\cdot y\cdot z) \leq$
2	66						1909	67
3	42						2304	42
5	30	2	0	0	1	1	3417	30
7	24	3	1	0	−1	1	2391	24
11	18	5	0	2	−2	1	1443	18
13	18	6	0	−1	−1	2	3196	18

Hence $m(x\cdot y\cdot z) \leq 67$. Next, we repeated the procedure with $X_0 = 67$, and μ as in the following table. We found

p	μ	γ_0^*	γ_1^*	γ_2^*	γ_3^*	γ_4^*	$\ell(\Gamma_\mu^*) >$	$ord_p(x\cdot y\cdot z) \leq$
2	55						364	56
3	35						301	35
5	25	2	1	1	1	0	622	25
7	20	3	1	−1	1	0	693	20
11	15	5	1	2	−2	−2	192	15
13	15	6	1	0	3	2	658	15

Hence $m(x\cdot y\cdot z) \leq 56$.

To find the solutions of (6.2) with $\mathrm{ord}_p(x \cdot y \cdot z)$ below the bounds given in the above table, we followed the following procedure. Suppose that we are at a certain moment interested in finding the solutions with $\mathrm{ord}_p(x \cdot y \cdot z) \le f(p)$ where $f(p)$ is given for $p = 2, \ldots, 13$. Choose p , and $\mu < f(p) - \mu_0$, and consider the lattice Γ_μ^* for these values of p, μ . If a solution x, y, z of (6.2) exists with $\mathrm{ord}_p(z) \ge \mu + \mu_0$, then the vector $[x_1, \ldots, x_4, x_0]^T$ with $x_i = \mathrm{ord}_{p_i}(x/y)$ for $i = 0, \ldots, 4$, is in the lattice. Its length is bounded by $\sqrt{(f(p_0)^2 + \ldots + f(p_4)^2)}$. All vectors in Γ_μ^* with length below this bound can be computed by the algorithm of Fincke and Pohst, as given in Section 3.6. Then all solutions of (6.2) corresponding to lattice points can be selected. Then we replace $f(p)$ by $\mu + \mu_0 - 1$, and we repeat the procedure for newly chosen p, μ .

We performed this procedure, starting with the bounds for $\mathrm{ord}_p(x \cdot y \cdot z)$ given in the above table for $f(p)$, and with p, m as in the table on the next page. Here, # stands for the number of solutions of (5.2) found at that stage. At the end we have $f(2) = 4$, $f(p) = 1$ for $p = 3, \ldots, 13$. The remaining solutions can be found by hand. □

<u>Remark.</u> Theorems 6.2 and 6.3 have applications in group theory (cf. Alex [1976]). We use Theorem 6.3 in Section 7.2.

<u>Remark.</u> The computer calculations for the proof of Theorem 6.3 took 438 sec., of which 412 were used for the first reduction step. In this first step we applied the L^3-algorithm in 11 steps (cf. Section 3.5), which cost on average about 60 sec. per lattice. The remaining 50 sec. were mainly used for the computation of the 24 $\vartheta_i^{(\mu)}$'s .

6.6. Examples related to the abc-conjecture.

Let x, y, z be positive integers. Put

$$G = \prod_{\substack{p | xyz \\ p \text{ prime}}} p .$$

For all x, y, z with $(x,y) = 1$ and $x + y = z$ we define

$$c(x,y,z) = \log z / \log G .$$

Recently, Oesterlé posed the problem to decide whether there exists an

p	m	#	p	m	#	p	m	#
2	44	–	2	13	1	2	10	2
3	28	–	2	12	2	2	9	3
5	20	–	2	11	2	2	8	6
7	16	–	3	13	–	2	7	15
11	12	–	3	12	–	2	6	16
13	12	–	3	11	–	2	5	26
2	33	–	3	10	1	2	4	31
3	21	–	3	9	1	2	3	44
5	15	–	3	8	1	3	6	5
7	12	–	3	7	6	3	5	8
11	9	–	5	9	–	3	4	16
13	9	–	5	8	–	3	3	35
2	22	–	5	7	–	3	2	54
3	14	–	5	6	–	3	1	87
5	10	–	5	5	6	5	4	1
7	8	–	7	7	–	5	3	5
11	6	–	7	6	–	5	2	18
13	6	–	7	5	1	5	1	36
2	21	–	7	4	4	7	3	–
2	20	–	11	5	–	7	2	6
2	19	–	11	4	1	7	1	18
2	18	–	11	3	4	11	2	1
2	17	–	13	5	–	11	1	8
2	16	–	13	4	–	13	2	–
2	15	–	13	3	1	13	1	4
2	14	–						

absolute constant C such that $c(x,y,z) < C$ for all x, y, z. Masser conjectured the stronger assertion that $c(x,y,z) < 1 + \epsilon$, when z exceeds some bound depending on ϵ only, for all $\epsilon > 0$. For a survey of related results and conjectures, see Stewart and Tijdeman [1986] and Vojta [1987].

It might be interesting to have some empirical results on $c(x,y,z)$, and to search for x, y, z for which it is large. From the preceding sections it may be clear that such x, y, z correspond to relatively short vectors in appropriate p-adic approximation lattices.

As a byproduct of the proofs of Theorems 5.5 and 6.3 we computed the value of $c(x,y,z)$, corresponding to many short vectors that we came across in performing the algorithm of Fincke and Pohst. All examples that we found with $c(x,y,z) \geq 1.4$ are listed below. Our search was rather unsystematic, so we do not guarantee that this list is complete in any sense. The largest value for $c(x,y,z)$ that occured is 1.626 , which was reached by

$$x = 11^2 = 121, \ y = 3^2 \cdot 5^6 \cdot 7^3 = 48234375, \ z = 2^{21} \cdot 23 = 48234496 .$$

This example was found on September 20, 1985, and has not yet been beaten, to the author's knowledge.

x	y	z	c(x,y,z)
11^2	$3^2 \cdot 5^6 \cdot 7^3$	$2^{21} \cdot 23$	1.62599
1	$2 \cdot 3^7$	$5^4 \cdot 7$	1.56789
7^3	3^{10}	$2^{11} \cdot 29$	1.54708
$5^2 \cdot 7937$	7^{13}	$2^{18} \cdot 3^7 \cdot 13^2$	1.49762
11^2	$3^9 \cdot 13$	$2^{11} \cdot 5^3$	1.48887
37	2^{15}	$3^8 \cdot 5$	1.48291
$2^7 \cdot 5^2$	$7^6 \cdot 41$	13^6	1.46192
1	$2^5 \cdot 3 \cdot 5^2$	7^4	1.45567
$2^{19} \cdot 13 \cdot 103$	7^{11}	$3^{11} \cdot 5^3 \cdot 11^2$	1.45261
1	$2^{12} \cdot 5^3$	$3^5 \cdot 7^2 \cdot 43$	1.44331
1	$2^4 \cdot 3^7 \cdot 547$	$5^8 \cdot 7^2$	1.43906
$2^{10} \cdot 7$	5^7	$3^8 \cdot 13$	1.43501
3	5^3	2^7	1.42657
5	3^{11}	$2^{10} \cdot 173$	1.41268

These results do not seem to yield any heuristical evidence for the truth or falsity of the abc-conjecture.

6.7. Tables.

Table I. (Theorem 6.2.)

$p=2,\ p_0=3,\ p_1=5$

x_0	$p_0^{x_0}$	x_1	$p_1^{x_1}$	sign	u	w
2	9	10	9765625	−1	4	−610351
10	59049	10	9765625	−1	4	−606661
4	81	12	244140625	−1	9	−476837
6	729	10	9765625	−1	5	−305153
2	9	8	390625	−1	3	−48827
6	729	8	390625	−1	3	−48737
10	59049	8	390625	−1	3	−41447
14	4782969	10	9765625	−1	7	−38927
4	81	8	390625	−1	4	−24409
0	1	8	390625	−1	5	−12207
8	6561	8	390625	−1	6	−6001
0	1	6	15625	−1	3	−1953
4	81	6	15625	−1	3	−1943
8	6561	6	15625	−1	3	−1133
6	729	6	15625	−1	4	−931
2	9	4	625	−1	3	−77
2	9	6	15625	−1	8	−61
0	1	4	625	−1	4	−39
4	81	4	625	−1	5	−17
0	1	2	25	−1	3	−3
2	9	2	25	−1	4	−1
1	3	1	5	1	3	1
1	3	3	125	1	7	1
2	9	0	1	−1	3	1
3	27	1	5	1	5	1
4	81	0	1	−1	4	5
4	81	2	25	−1	3	7
6	729	2	25	−1	6	11
6	729	4	625	−1	3	13
3	27	3	125	1	3	19
5	243	3	125	1	4	23
5	243	1	5	1	3	31
7	2187	5	3125	1	6	83
6	729	0	1	−1	3	91
7	2187	1	5	1	4	137
11	177147	1	5	1	10	173
3	27	5	3125	1	4	197
8	6561	0	1	−1	5	205
7	2187	3	125	1	3	289
8	6561	4	625	−1	4	371

Table continued

Table I. (cont.)

x_0	$p_0^{x_0}$	x_1	$p_1^{x_1}$	sign	u	w
1	3	5	3125	1	3	391
5	243	5	3125	1	3	421
9	19683	3	125	1	5	619
8	6561	2	25	−1	3	817
10	59049	6	15625	−1	5	1357
5	243	7	78125	1	5	2449
9	19683	1	5	1	3	2461
9	19683	5	3125	1	3	2851
10	59049	2	25	−1	4	3689
12	531441	4	625	−1	7	4147
1	3	7	78125	1	4	4883
9	19683	7	78125	1	4	6113
13	1594323	7	78125	1	8	6533
10	59049	4	625	−1	3	7303
10	59049	0	1	−1	3	7381
12	531441	8	390625	−1	4	8801
3	27	7	78125	1	3	9769
7	2187	7	78125	1	3	10039
11	177147	5	3125	1	4	11267
3	27	9	1953125	1	7	15259
11	177147	3	125	1	3	22159
11	177147	7	78125	1	3	31909
12	531441	0	1	−1	4	33215
12	531441	6	15625	−1	3	64477
12	531441	2	25	−1	3	66427
11	177147	9	1953125	1	5	66571
13	1594323	3	125	1	4	99653
7	2187	9	1953125	1	4	122207
14	4782969	2	25	−1	5	149467
13	1594323	1	5	1	3	199291
13	1594323	5	3125	1	3	199681
1	3	9	1953125	1	3	244141
5	243	9	1953125	1	3	244171
9	19683	9	1953125	1	3	246601
14	4782969	6	15625	−1	4	297959
13	1594323	9	1953125	1	3	443431
15	14348907	5	3125	1	5	448501
14	4782969	8	390625	−1	3	549043
14	4782969	4	625	−1	3	597793
14	4782969	0	1	−1	3	597871
16	43046721	0	1	−1	6	672605
9	19683	11	48828125	1	6	763247
15	14348907	1	5	1	4	896807

Table continued

Table I. (cont.)

$p=3,\ p_0=2,\ p_1=5$

x_0	$p_0^{x_0}$	x_1	$p_1^{x_1}$	sign	u	w
14	16384	10	9765625	−1	4	−120361
9	512	9	1953125	−1	3	−72319
4	16	8	390625	−1	3	−14467
12	4096	6	15625	−1	3	−427
7	128	5	3125	−1	4	−37
2	4	4	625	−1	3	−23
1	2	2	25	1	3	1
5	32	1	5	−1	3	1
6	64	3	125	1	3	7
11	2048	4	625	1	5	11
9	512	0	1	1	3	19
10	1024	2	25	−1	3	37
3	8	6	15625	1	4	193
15	32768	3	125	−1	4	403
14	16384	1	5	1	3	607
17	131072	7	78125	−1	3	1961
16	65536	5	3125	1	3	2543
8	256	7	78125	1	3	2903
19	524288	2	25	1	4	6473
18	262144	0	1	−1	3	9709
23	8388608	1	5	−1	6	11507
13	8192	8	390625	1	3	14771
22	4194304	8	390625	−1	5	15653
10	1024	11	48828125	1	7	22327
18	262144	9	1953125	1	4	27349
20	1048576	4	625	−1	3	38813
0	1	9	1953125	1	3	72338
21	2097152	6	15625	1	3	78251
5	32	10	9765625	1	3	361691
24	16777216	3	125	1	3	621383
23	8388608	10	9765625	1	3	672379
26	67108864	7	78125	1	4	829469

$p=5,\ p_0=2,\ p_1=3$

x_0	$p_0^{x_0}$	x_1	$p_1^{x_1}$	sign	u	w
12	4096	16	43046721	−1	3	−344341
5	32	15	14348907	−1	3	−114791
7	128	1	3	−1	3	1
6	64	8	6561	1	3	53
14	16384	2	9	−1	3	131
13	8192	9	19683	1	3	223
20	1048576	10	59049	1	3	8861
21	2097152	3	27	−1	3	16777

Table II. (Theorem 6.3.)

			ord$_p(x)$						ord$_p(y)$						ord$_p(z)$					
x	y	z	$p=2$	3	5	7	11	13	$p=2$	3	5	7	11	13	$p=2$	3	5	7	11	13
2401	4160	6561	0	0	0	4	0	0	6	0	1	0	0	1	0	8	0	0	0	0
875	6561	7436	0	0	3	1	0	0	0	8	0	0	0	0	2	0	0	0	1	2
1183	6561	7744	0	0	0	1	0	2	0	8	0	0	0	0	6	0	0	0	2	0
1125	8192	9317	0	2	3	0	0	0	13	0	0	0	0	0	0	0	0	1	3	0
1183	8192	9375	0	0	0	1	0	2	13	0	0	0	0	0	0	1	5	0	0	0
16	14625	14641	4	0	0	0	0	0	0	2	3	0	0	1	0	0	0	0	4	0
81	14560	14641	0	4	0	0	0	0	5	0	1	1	0	1	0	0	0	0	4	0
1936	13689	15625	4	0	0	0	2	0	0	4	0	0	0	2	0	0	6	0	0	0
3718	11907	15625	1	0	0	0	1	2	0	5	0	2	0	0	0	0	6	0	0	0
5824	9801	15625	6	0	0	1	0	1	0	4	0	0	2	0	0	0	6	0	0	0
49	16335	16384	0	0	0	2	0	0	0	3	1	0	2	0	14	0	0	0	0	0
2695	13689	16384	0	0	1	2	1	0	0	4	0	0	0	2	14	0	0	0	0	0
8019	8788	16807	0	6	0	0	1	0	2	0	0	0	0	3	0	0	0	5	0	0
3584	14641	18225	9	0	0	1	0	0	0	0	0	0	4	0	0	6	2	0	0	0
1625	16807	18432	0	0	3	0	0	1	0	0	0	5	0	0	11	2	0	0	0	0
3993	16807	20800	0	1	0	0	3	0	0	0	0	5	0	0	6	0	2	0	0	1
49	28512	28561	0	0	0	2	0	0	5	4	0	0	1	0	0	0	0	0	0	4
12936	15625	28561	3	1	0	2	1	0	0	0	6	0	0	0	0	0	0	0	0	4
22000	6561	28561	4	0	3	0	1	0	0	8	0	0	0	0	0	0	0	0	0	4
15625	17303	32928	0	0	6	0	0	0	0	0	0	0	3	1	5	1	0	3	0	0
507	32768	33275	0	1	0	0	0	2	15	0	0	0	0	0	0	0	2	0	3	0
10985	41503	52488	0	0	1	0	0	3	0	0	0	3	2	0	3	8	0	0	0	0
10000	49049	59049	4	0	4	0	0	0	0	0	0	3	1	1	0	10	0	0	0	0
14641	46875	61516	0	0	0	0	4	0	0	1	6	0	0	0	2	0	0	1	0	3
7168	78125	85293	10	0	0	1	0	0	0	0	7	0	0	0	0	8	0	0	0	1
20449	97200	117649	0	0	0	0	2	2	4	5	2	0	0	0	0	0	0	6	0	0
13	151250	151263	0	0	0	0	0	1	1	0	4	0	2	0	0	2	0	5	0	0
12005	161051	173056	0	0	1	4	0	0	0	0	0	0	5	0	10	0	0	0	0	2
121	255879	256000	0	0	0	0	2	0	0	9	0	0	0	1	11	0	3	0	0	0
2197	583443	585640	0	0	0	0	0	3	0	5	0	4	0	0	3	0	1	0	4	0
91	1771470	1771561	0	0	0	1	0	1	1	11	1	0	0	0	0	0	0	0	6	0

Table III.

$-\log_2 5 / \log_2 3 =$

```
0.10101 11101 00001 11110 11000 10101 00000 01001 11101 00010 10000 10011 10110 10000 01011 11100 00001 11010 00000 00001
  00010 11100 11100 10111 01001 01101 11000 01010 01110 01010 11110 00000 00101 00110 00100 00011 01111 01110 01010 11101
  10010 01001 00001 10100 00111 00001 11111 01111 00011 10110 00000 00101 01101 01100 00010 01110 11100 11101 11011 01011
  10111 00100 11111 00100 00100 10001 10010 01011 10010 01101 00000 01100 01111 10111 01101 00110 11110 00000 01100 11010
  00001 00111 11011 11001 01000 10000 00110 11011 11001 01000 01110 11001 00010 00010 11000 01000 00110 01100 11101 11110
  00000 11101 10101 11100 10010 11101 01011 10001 01100 11000 01100 00110 01001 01100 01000 01001 10001 11000 01101 10100
  00110 00011 0111....
```

$-\log_2 7 / \log_2 3 =$

```
0.01001 01011 01111 11100 11010 01111 11111 10010 01000 11100 00011 00100 01011 11110 00101 11010 11110 10001 00000 01110
  01011 00101 10010 00111 10111 01001 10001 11000 11011 10111 01011 01001 01011 00100 11001 11000 10111 01100 00001 01110
  11000 01011 00011 01110 10000 01101 11000 11001 00010 00011 01100 11110 11010 11110 00001 10110 01010 00001 01000 11011
  10010 01000 00011 01011 10010 11001 10000 01101 10111 01001 01000 00101 00011 11001 10000 00011 00101 00100 00000 10011
  11100 01110 11110 10101 00101 01110 11100 10000 01011 00100 01100 11100 00100 01110 10001 00001 10000 00111 10111 00011
  10111 01100 10110 00111 00101 10011 01100 11111 00101 01011 11101 11011 01011 10110 01100 10001 11011 01100 01001 11111
  10001 11110 1010....
```

$-\log_2 11 / \log_2 3 =$

```
0.10011 01110 00001 01001 00110 01010 01110 00100 00101 10000 01000 11001 10010 01111 10110 00011 00101 01110 00011 11101
  10100 01101 01101 10111 10110 01100 10110 11110 00000 11000 01000 11111 10010 10011 10110 01001 01111 00101 00011 11100
  01100 10100 01000 10101 00010 01011 10111 10000 00001 01000 11010 01010 11010 11100 00001 10110 01001 11111 11110 10010
  11101 11100 10010 00100 11000 00000 01110 10100 00101 10010 10010 10010 01111 10100 01001 10111 01000 10101 01110 01000
  11001 11001 10011 11110 10000 11001 10110 10101 00010 11100 11101 11011 10110 00111 11010 11000 01000 10111 11110 10010
  10101 11100 11011 00111 11000 00111 11100 01101 11100 01101 00001 00100 01001 11000 01101 10101 00110 10101 00001 00001
  10101 11001 1111....
```

$-\log_2 13 / \log_2 3 =$

```
0.11011 10110 10100 10001 01100 01111 10001 00110 01110 00001 01110 01011 11101 10110 11100 00111 11111 11001 11011 11110
  00011 11110 01000 10010 11011 11101 01000 11000 11111 01011 00001 10101 00110 00000 00001 01101 10010 00101 00100 10101
  01011 11100 10011 11011 00000 01110 10100 00011 10000 11101 00010 00000 10011 01011 11011 10000 11010 10010 11010 00100
  10010 00001 01000 10101 10110 01001 01000 11111 01101 00111 11100 10001 01110 11000 01010 10011 11001 01111 01111 00001
  10011 00000 00101 10101 10111 11100 01011 01011 11000 10001 11110 01000 10110 00011 11011 11011 11010 11110 10100 01110
  01000 00111 00000 00000 00011 10111 11111 10100 01111 10010 01100 10101 00111 10110 11010 01111 10000 11101 10111 10001
  00000 01100 1001....
```

<u>Table III.</u> (cont.)

$-\log_3 5 / \log_3 2 =$

```
0.11022 12121 22001 12010 21102 10210 10022 20212 20010 10112 22201 21021 21022 10000 22020 12012 02022 21001 00012 02020
  21210 12202 12200 00000 10120 00211 12021 10120 02100 10222 22122 01201 21111 11121 11001 20222 10000 20121 22221 01002
  20220 12211 22211 00100 20202 00012 11112 10122 21001 21200 12201 12220 11100 01102 20010 11102 10222 00020 21202 21112
  20201 21100 11212 22222 21120 02020 12121 02122 11111 10001 10220 21022 10012 11212 20001 10211 02120 02122 1....
```

$-\log_3 7 / \log_3 2 =$

```
0.20101 10202 20011 12121 01102 11100 01210 20120 02122 02012 20202 00121 21200 01201 11120 11211 11212 22100 00100 22201
  20021 11112 01122 00011 22100 00000 22011 11100 12010 22110 12122 00222 10220 21102 20001 02101 00121 11002 11012 12201
  21011 20100 01110 02000 21222 12010 02201 22012 01022 02021 00210 10221 00221 20202 02222 22122 00100 12021 21220 02220
  20000 00002 00111 11221 11002 20102 12212 12012 22122 00211 01210 01102 21010 20121 01020 11111 20002 10122 2....
```

$-\log_3 11 / \log_3 2 =$

```
0.21112 20101 00222 20222 01212 01100 12100 01201 01111 01212 01210 20121 20001 12021 01122 21202 12020 00212 11102 11002
  01001 10200 22202 02001 20022 10221 00010 10011 22220 01021 02121 00211 22210 21101 22012 11111 02010 00221 00102 20111
  20202 01201 01220 22022 11221 10121 10202 10011 11002 10220 22110 21121 00112 02122 21200 01021 21002 21002 10010 00110
  00101 12202 12000 21012 11010 11020 00222 00012 11201 11010 00122 01120 22200 20112 12122 10202 01211 00210 2....
```

$-\log_3 13 / \log_3 2 =$

```
0.10221 02211 12122 22010 10002 01221 00121 02020 11201 02021 02112 21010 20122 02001 02112 21012 10222 01002 01200 01211
  10111 21100 12121 11010 02000 02212 11111 21220 22020 02000 01222 12112 02100 10110 20002 10222 02112 20112 11100 00211
  20012 11102 22220 00112 00001 11110 11102 22201 01122 22211 22201 11011 22201 01200 22121 02101 22222 22002 01010 01021
  12020 20111 12102 00011 02002 02000 10211 00222 12202 02202 20212 22012 01222 20220 11211 20021 11111 00000 2....
```

$-\log_5 3 / \log_5 2 =$

```
0.33002 02003 04411 23120 44012 01011 00044 43204 30340 00023 14333 12413 43420 40302 10202 44104 32433 24432 03021 12311
  34044 40231 04112 33230 00242 14232 14400 31104 42112 44033 11014 44344 12114 44211 32120 43131 34041 00411 34233 41410
  24120 42032 43014 21421 40044 01142 21004 42021 14011 10404 00214 31110 04441 42431 24423 0243....
```

$-\log_5 7 / \log_5 2 =$

```
0.03044 34433 10114 43203 12033 14002 12341 31312 03421 00343 41423 00040 24241 22103 14240 32214 11401 42230 13040 33404
  04310 43034 13233 23241 43002 44411 41124 22443 42412 30420 11223 43101 01000 42112 10443 34210 03410 14414 02220 24443
  13332 33123 23331 20323 44440 13210 14403 32122 03040 31123 04212 22443 44223 23133 02003 1240....
```

Table III. (cont.)

$-\log_5 11 / \log_5 2 =$

0.44032 21012 13124 21134 03320 33422 21041 12112 42420 00220 41143 12040 32144 21100 01304 24013 43401 23313 12022 34404
12413 10214 30123 11014 24110 42444 42030 02413 20241 22304 23423 13414 03234 30000 10334 44322 00330 01104 44410 44113
31022 33142 14441 44113 21413 23132 31413 32032 01221 40210 24101 30133 13110 13400 22110 2334....

$-\log_5 13 / \log_5 2 =$

0.12423 02224 01323 24314 23021 32420 141[illegible]4 41224 04403 11334 43213 33303 03130 32244 11133 43243 23422 11320 41041 31134
41320 34110 03024 40012 23213 10014 41441 04420 40114 00021 33224 30103 03243 32031 22021 20234 32441 00013 04203 43134
22012 30332 22422 43110 13302 34431 13241 13230 44204 14432 33210 24121 13144 03230 14301 3040....

$-\log_7 2 / \log_7 3 =$

0.20603 14521 11264 52354 4[illegible]764 60036 13315 13044 46363 40432 02366 04135 21304 53356 32205 44546 66301 00123 63633 04024
40631 53556 64053 50031 2[illegible]36 46625 03465 02235 11551 46123 25164 52364 25520 12240 64220 00164 43634 02066 41264 61233
41326 32413 65633 52502 526....

$-\log_7 5 / \log_7 3 =$

0.62250 35002 24045 66544 01041 43506 34535 04453 26545 45453 33261 65353 53330 22443 10105 55005 62520 33063 16320 22253
42306 51054 13301 06465 43020 41555 41121 64255 11350 55053 64515 44465 36222 25605 66346 16142 31340 45522 31033 14255
21343 24510 62633 00155 361....

$-\log_7 11 / \log_7 3 =$

0.25035 56505 33553 02331 10224 32143 50543 02561 42352 23430 26326 53446 23462 31210 02416 02335 13066 54240 13006 60451
00441 56630 66142 56540 44042 52255 15314 10131 40626 10543 02504 04254 45066 50232 65102 41555 41254 02211 54156 34054
41366 64215 64014 56645 550....

$-\log_7 13 / \log_7 3 =$

0.21305 11055 56501 22565 55610 32506 13150 66465 56420 46465 21650 16426 11613 41010 66512 26566 50652 020[illegible] 40431 56566
43655 56120 34610 53642 61544 36122 61225 42410 23035 15004 26220 14444 23632 33426 15605 51104 34116 04520 65502 65542
24255 36603 45452 66563 536....

Table III. (cont.)

$-\log_{11} 3 / \log_{11} 2 =$

0.08248 A4245 06166 43468 58202 44A56 73171 16758 A203A 8A543 28431 86731 11411 4A296 993A7 31A79 00421 95444 80670 57433
59439 78064 34745 1A710 64682 08044 81761 27049 03452 3661A 40979 29601 898A4 50....

$-\log_{11} 5 / \log_{11} 2 =$

0.351A9 7223A 31378 09193 42445 306A3 96588 11862 48667 AA6A2 39A03 77139 01693 21678 33652 12687 95AA8 24190 78276 28711
08399 68022 2A607 55A17 2231A 80798 76947 73936 21835 30A1A 95324 1A8A3 82999 67....

$-\log_{11} 7 / \log_{11} 2 =$

0.44804 92167 71327 83472 37453 00781 3256A 2A367 85671 88907 799A1 4AAAA 784A1 29329 A6950 17481 86846 17379 94130 77091
29354 33161 9A146 03746 52A14 20214 22541 58A91 50337 7954A 89A01 43809 A8152 52....

$-\log_{11} 13 / \log_{11} 2 =$

0.9011A 94962 52990 39096 3A68A 7556A 1A4A3 44758 57692 20188 42770 072A3 9A977 8819A 97518 14396 07360 899A2 99391 26176
84077 81181 54473 58532 58A01 91643 28056 63940 99265 27989 37450 85913 91289 56....

$-\log_{13} 3 / \log_{13} 2 =$

0.621B3 15581 0A077 3B5C8 49202 39A32 82105 848C7 70988 B863B 75151 52114 5C25A 04902 6B6C7 377B9 3122B 5CAC0 13945 A2471
9B4BA A79C2 7A91C 5A989 C392A CC16A A20A1 75C6B 06BB6 8A3B9 C782B AA70C AB218 C9....

$-\log_{13} 5 / \log_{13} 2 =$

0.44570 79C51 73665 3796C B7C61 335A0 79906 2B429 51211 4900B 481B1 621AB 2AC77 C2291 1662A BB03A 8CB9C 77331 74992 11C07
BB101 10301 77310 B8B28 83AB2 57975 7C697 57928 23B72 297CB 0A414 32B3C A67A8 48....

$-\log_{13} 7 / \log_{13} 2 =$

0.A1C78 9C71A 63110 51424 42CA9 0AAA7 B225B B0281 501B1 976C2 3C05B 09CA3 AB8[illegible]3 C3251 838AC 72502 A1844 03603 644A8 A8501
173BB BBC1C 30466 223C6 C98B4 564C2 47140 28856 C8676 15C30 12892 A3317 163C8 CA....

$-\log_{13} 11 / \log_{13} 2 =$

0.1760A A080C 20874 BB876 B2162 75989 CB19B B7CC2 26BB7 87093 5A833 A9375 AB4BA 8C0BC 1A698 96C6B A9411 34B75 4B718 63BC3
571A9 14566 8819B A4B95 B4244 452A8 29623 49AA5 CB804 AC61A CC513 08855 79185 43....